POLE PLACEMENT CONTROL

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Contents

Summary

Pole placement by output feedback is separated into pole placement by state feedback and observer pole placement. Since both problems are dual, only the state feedback case is worked out in detail. In the single-input case, the pole placement problem has a unique solution. It is found efficiently by Ackermann's formula. A numerically stable evaluation via Hessenberg form is shown.

In the multi-input case, the solution to the pole placement problem is non-unique. Therefore other specifications in addition to pole placement can be satisfied. Such choices are limited only by feedback invariants. These invariants are exhibited in a Brunovski canonical form, which is fully characterized by a set of integers, the controllability indices (also called Kronecker indices). A feedback transformation to this form also provides a deadbeat solution with the smallest number of sampling intervals until each state variable comes to complete rest. Going backwards in the same steps, new life, i.e., new dynamics of subsystems and new couplings between subsystems can be given to the deadbeat structure. Finally, the total feedback matrix is composed as a sum of a deadbeat feedback matrix and a revival feedback matrix. The overall calculations may be done in a numerically stable way by transformation to HN form.

1. Introduction

Consider a system with linear state-space model

$$\begin{aligned}\dot{\boldsymbol{x}} &= \boldsymbol{A}\,\boldsymbol{x} + \boldsymbol{B}\,\boldsymbol{u} \\ \boldsymbol{y} &= \boldsymbol{C}\,\boldsymbol{x}\end{aligned} \tag{1}$$

The dimensions of the vectors are:
input vector dim $\boldsymbol{u} = m$

state vector dim $\boldsymbol{x} = n$

output vector dim $\boldsymbol{y} = p$

A, B and ***C*** are real matrices of appropriate dimensions.

Application of the Laplace transform to Eq. (1) yields the input-output description

$$\boldsymbol{y}(s) = \boldsymbol{C}(s\,\boldsymbol{I} - \boldsymbol{A})^{-1}\,\boldsymbol{B}\,\boldsymbol{u}(s) \tag{2}$$

It has the form of a $p \times m$ matrix of rational transfer functions. The uncancelled denominator

$$a(s) = \det(s\,\boldsymbol{I} - \boldsymbol{A}) \tag{3}$$

is the (open-loop) characteristic polynomial. Its roots are the eigenvalues of $\boldsymbol{A}$, they characterize the dynamics of the system (1).

If a factor $(s - s_k)$ in $a(s)$ is cancelled by the numerator of the transfer function from input j to output i (i.e., in the ij-element of the transfer matrix), then the eigenvalue s_1 is either not controllable from input j or not observable from output i (or both). Therefore the eigenvalue s_k cannot be shifted feedback from output i to input j. After execution of all cancellations the poles remain in the denominator, which can be shifted to an assigned position. Synonymous expressions for this process are “pole placement”, “pole assignment” and “pole shifting”.

For simplicity we assume, that the pair (***A, B***) is controllable, i.e., each eigenvalue is controllable form at least one input. Correspondingly it is assumed, that the pair (***A, C***) is observable, i.e., each eigenvalue is observable from at least one output.

A typical control problem arises, when the location of poles indicates an unstable or weakly damped or very slow response of the system. A better dynamic behavior can be achieved by feedback of $\boldsymbol{y}$ to $\boldsymbol{u}$ by a controller. Pole placement control is a systematic way to determine this controller such that the closed-loop system has a desired set of poles.

2. Separation of state observation and state feedback

A pole placement controller consists of an observer that generates an estimate $\hat{x}$ for the state x and a state feedback of $\hat{x}$ to $\boldsymbol{u}$. If all states are measured, i.e., rank $\boldsymbol{C}$ = n, then no observer is needed and $\boldsymbol{x}$ is fed back to $\boldsymbol{u}$. Assume in this section rank $\boldsymbol{C} = p < n$. An observer of order n may be written as

$$\dot{\hat{\boldsymbol{x}}} = \boldsymbol{A}\,\hat{\boldsymbol{x}} + \boldsymbol{B}\,\boldsymbol{u} + \boldsymbol{L}(\boldsymbol{y} - \boldsymbol{C}\,\hat{\boldsymbol{x}}) \tag{4}$$

Its state $\hat{\boldsymbol{x}}$ is fed back via

$$\boldsymbol{u} = \boldsymbol{V}\,\boldsymbol{r} - \boldsymbol{K}\,\hat{\boldsymbol{x}} \tag{5}$$

where r is the reference input, K. L and $\boldsymbol{V}$ are the free parameters in this controller structure. Introducing the estimation error $\tilde{x} = x - \hat{x}$, the state equation of the overall system described by Eqs. (1), (4) and (5) becomes

$$\begin{bmatrix} \dot{\boldsymbol{x}} \\ \dot{\tilde{\boldsymbol{x}}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A} - \boldsymbol{B}\boldsymbol{K} & \boldsymbol{B}\boldsymbol{K} \\ \boldsymbol{0} & \boldsymbol{A} - \boldsymbol{L}\boldsymbol{C} \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \tilde{\boldsymbol{x}} \end{bmatrix} + \begin{bmatrix} \boldsymbol{B} \\ \boldsymbol{0} \end{bmatrix} \boldsymbol{V}\,\boldsymbol{r} \tag{6}$$

The block triangular structure of the system matrix shows that the characteristic polynomial of the closed-loop system is the product of $p(s) = \det(s\boldsymbol{I} - \boldsymbol{A} + \boldsymbol{B}\boldsymbol{K})$ and $q(s) = \det(s\boldsymbol{I} - \boldsymbol{A} + \boldsymbol{L}\boldsymbol{C})$. This separation property allows the independent placement of the control poles in $p(s)$ by K and of the observer poles in $q(s)$ by L. The conclusion on separation also holds, if the full order observer is replaced by an observer of the reduced order $n - p$. An elegant result in linear control theory is that poles of a controllable and observable system can be assigned arbitrarily by linear state feedback.

The two pole placements are dual and can be made identical by transposing the second matrix as $q(s) = \det(s\boldsymbol{I} - \boldsymbol{A}^T + \boldsymbol{C}^T\,\boldsymbol{L}^T)$. The correspondences are for the given matrices $\boldsymbol{A} \rightarrow \boldsymbol{A}^T$, $\boldsymbol{B} \rightarrow \boldsymbol{C}^T$ and the polynomial equation is to be solved for $\boldsymbol{K} \rightarrow \boldsymbol{L}^T$. Therefore observer pole placement will not be discussed further in the following sections and only the equation $p(s) = \det(s\boldsymbol{I} - \boldsymbol{A} + \boldsymbol{B}\boldsymbol{K})$ will be treated as it applies also to full state feedback $\boldsymbol{u} = \boldsymbol{V}\,\boldsymbol{r} - \boldsymbol{K}\,\boldsymbol{x}$.

3. The single-input case

In the single-input case, $m = 1$, the feedback vector $\boldsymbol{k}^T$ has n elements and there are n assigned poles for the closed-loop. The resulting set of n equations in n unknowns has a unique solution if and only if the pair $(\boldsymbol{A}, \boldsymbol{b})$ is controllable, i.e.,

$\det[\boldsymbol{b}, \boldsymbol{Ab} \ldots \boldsymbol{A}^{n-1}\boldsymbol{b}] \neq 0$. Consider the equation $p(s) = \det(s\boldsymbol{I} - \boldsymbol{A} + \boldsymbol{b}\boldsymbol{k}^T)$ with a given controllable pair (***A, b***). In analyzing the effect of a given state feedback vector $\boldsymbol{k}$ on the roots of $p(s)$ one has to go through the numerical factorization of the polynomial $p(s)$. A symbolic solution is not possible except for a few very simple cases. In the opposite direction, i.e., given the roots of $p(s)$, find the required $\boldsymbol{k}$, a symbolic solution of the synthesis problem is possible. It will be presented in the next section.

a. Ackermann's formula

The solution of

$$\begin{aligned} &p(s) = \det(s\boldsymbol{I} - \boldsymbol{A} + \boldsymbol{b}\boldsymbol{k}^T) = \\ &p_0 + p_1 s + \ldots + p_{n-1}s^{n-1} + s^n, \\ &(\boldsymbol{A}, \boldsymbol{b}) \text{ controllable} \end{aligned} \tag{7}$$

for $\boldsymbol{k}^T$ will be derived in this section. Let $\boldsymbol{F} = \boldsymbol{A} - \boldsymbol{b}\boldsymbol{k}^T$, expand $\boldsymbol{F}^k$ into expressions of the form $\boldsymbol{A}^k$ and $\boldsymbol{A}^i\boldsymbol{b}\boldsymbol{k}^T\boldsymbol{F}^j$, $i + j = k - 1$, and evaluate

$$\begin{aligned} &p(\boldsymbol{F}) = p_0\boldsymbol{F}^0 + p_1\boldsymbol{F}^1 + \ldots \\ &+ p_{n-1}\boldsymbol{F}^{n-1} + \boldsymbol{F}^n. \end{aligned}$$

$$\begin{array}{ll|l} \boldsymbol{F}^0 = & \boldsymbol{A}^0 \quad (=\boldsymbol{I}) & \times p_0 \\ \boldsymbol{F}^1 = & \boldsymbol{A}^1 - \boldsymbol{b}\boldsymbol{k}^T & \times p_1 \\ \boldsymbol{F}^2 = & \boldsymbol{A}^2 - \boldsymbol{Ab}\boldsymbol{k}^T - \boldsymbol{b}\boldsymbol{k}^T\boldsymbol{F} & \times p_2 \\ \vdots & & \vdots \\ \boldsymbol{F}^n = & \boldsymbol{A}^n - \boldsymbol{A}^{n-1}\boldsymbol{b}\boldsymbol{k}^T - & \times 1 \\ & \boldsymbol{A}^{n-2}\boldsymbol{b}\boldsymbol{k}^T\boldsymbol{F} - \ldots - \boldsymbol{b}\boldsymbol{k}^T\boldsymbol{F}^{n-1} & \\ \hline \end{array}$$

$$p(\boldsymbol{F}) = p(\boldsymbol{A}) - \left[\boldsymbol{b}, \boldsymbol{Ab}, \ldots \boldsymbol{A}^{n-1}\boldsymbol{b}\right]\begin{bmatrix} \vdots \\ \boldsymbol{k}^T \end{bmatrix} \tag{8}$$

The polynomial $p(s) = \det(s\boldsymbol{I} - \boldsymbol{F})$ is the characteristic polynomial that shall be given to $\boldsymbol{F}$, therefore, by the CAYLEY-HAMILTON theorem, $p(\boldsymbol{F}) = \boldsymbol{0}$ and $\begin{bmatrix} \vdots \\ \boldsymbol{k}^T \end{bmatrix} = \left[\boldsymbol{b}, \boldsymbol{Ab}, \dots \boldsymbol{A}^{n\text{-}1}\boldsymbol{b}\right]^{-1} p(\boldsymbol{A})$. From the last row of this equation follows

$$\boldsymbol{k}^T = \boldsymbol{e}^T\ p(\boldsymbol{A}) \tag{9}$$

where $\boldsymbol{e}^T = \begin{bmatrix} 0 & \dots & 0 & 1 \end{bmatrix} \left[\boldsymbol{b},\ \boldsymbol{Ab},\ \dots\ \boldsymbol{A}^{n-1}\boldsymbol{b}\right]^{-1}$

is the last row of the inverted controllability matrix (non singular by the assumption of controllability). Eq. (9) is known as Ackermann's formula. Note that $p(s)$ may be given in factorized form as

$$p(s) = \begin{cases} \prod_{i=1}^{n/2} (s^2 + b_i\, s + c_i) \\ \qquad\qquad\qquad \text{for } n \text{ even} \\ (s+d) \prod_{i=1}^{(n-1)/2} (s^2 + b_i\, s + c_i) \\ \qquad\qquad\qquad \text{for } n \text{ odd} \end{cases} \tag{10}$$

rather than the multiplied form Eq. (7). The closed-loop system is then stable if and only if all b_i, c_i and d are positive. This is a characterization of all stabilizing state feedback gains $\boldsymbol{k}$ in Eq. (9).

The factorized form is useful if the design is performed in consecutive steps, where in each step only two eigenvalues are shifted. Assume the open-loop characteristic polynomial Eq. (3) is factorized as

$$a(s) = a_{\text{inv}}(s) \cdot (a_0 + a_1\, s + s^2) \tag{11}$$

where $a_{\text{inv}}(s)$ is a polynomial of degree $n-2$, whose roots shall remain unchanged in a design step. Then the closed-loop characteristic polynomial is specified as

$$p(s) = a_{\text{inv}}(s)\,(b_0 + b_1\, s + s^2) \tag{12}$$

and Eq. (9) reads

$$\boldsymbol{k}^T = \boldsymbol{e}_{inv}{}^T\ (b_0\boldsymbol{I} + b_1\,\boldsymbol{A} + \boldsymbol{A}^2\ ) \tag{13}$$

where $\boldsymbol{e}_{inv}{}^T = \boldsymbol{e}^T a_{inv}(\boldsymbol{A})$. Since $\boldsymbol{0}^T = \boldsymbol{e}_{\text{inv}}{}^T (a_0 \boldsymbol{I} + a_1 \boldsymbol{A} + \boldsymbol{A}^2)$, Eq. (13) may be written as

$$\boldsymbol{k}^T = \boldsymbol{e}_{inv}{}^T \left[(b_0 - a_0) \boldsymbol{I} + (b_1 - a_1) \boldsymbol{A} \right] \tag{14}$$

The two vectors $\boldsymbol{e}_{inv}{}^T$ and $\boldsymbol{e}_{inv}{}^T \boldsymbol{A}$ span a linear subspace of the $\boldsymbol{K}$-space, in which the $n-2$ open-loop poles contained in $a_{\text{inv}}(s)$, are not observable and cannot be shifted.

The factorized form of Eq. (9) is also useful to determine the sensitivity of the state-feedback vector $\boldsymbol{k}^T$ with respect to the placement of one or two eigenvalues.

For a real eigenvalue at $s = s_1$

$$\boldsymbol{k}^T = \boldsymbol{e}^T (\boldsymbol{A} - s_1 \boldsymbol{I}) \, r(\boldsymbol{A})$$

where $r(s)$ contains the remaining $n-1$ eigenvalues. Then

$$\frac{\partial \boldsymbol{k}^T}{\partial s_1} = -\boldsymbol{e}^T r(\boldsymbol{A}) \tag{15}$$

For a pair of eigenvalues at the roots of $b_0 + b_1 s + s^2$

$$\begin{gathered} \boldsymbol{k}^T = \boldsymbol{e}^T (b_0 \boldsymbol{I} + b_1 \boldsymbol{A} + \boldsymbol{A}^2) \, r(\boldsymbol{A}) \\ \frac{\partial \boldsymbol{k}^T}{\partial b_0} = \boldsymbol{e}^T r(\boldsymbol{A}), \quad \frac{\partial \boldsymbol{k}^T}{\partial b_1} = \boldsymbol{e}^T \boldsymbol{A} \; r(\boldsymbol{A}) \end{gathered} \tag{16}$$

b. Numerically stable calculation via Hessenberg form

Eq. (9) may be written as

$$\begin{aligned} \boldsymbol{k}^T &= \boldsymbol{e}^T (p_0 \boldsymbol{I} + p_1 \boldsymbol{A} + \ldots \\ &\qquad + p_{n-1} \boldsymbol{A}^{\boldsymbol{n-1}} + \boldsymbol{A}^{\boldsymbol{n-1}}) \\ &= [\, p_0 \;\; p_1 \ldots p_{n-1} \;\; 1 \,] \, \boldsymbol{E} \end{aligned} \tag{17}$$

The matrix

$$\boldsymbol{E} = \begin{bmatrix} \boldsymbol{e}^T \\ \boldsymbol{e}^T \boldsymbol{A} \\ \vdots \\ \boldsymbol{e}^T \boldsymbol{A}^{\boldsymbol{n}} \end{bmatrix}$$

is called pole placement matrix. The form Eq. (17) illustrates that it is not necessary to evaluate $p(\boldsymbol{A})$ by calculations with n^2-matrices. The calculation of $\boldsymbol{E}$ only requires operations on n-vectors.

A numerically stable way of computing $\boldsymbol{e}^T$, the last row of the inverted controllability matrix, is via transformation of the pair $(\boldsymbol{A}, \boldsymbol{b})$ to Hessenberg form with

$$\boldsymbol{A_H} = \boldsymbol{T}_H \boldsymbol{A} \boldsymbol{T}_H{}^{-1}, \; \boldsymbol{b}_H = \boldsymbol{T}_H \boldsymbol{b}$$

$$\left[\boldsymbol{A_H} \; \vdots \; \boldsymbol{b}_H\right] = \left[\begin{array}{ccccc|c} x & \otimes & 0 & \dots & & 0 \\ x & x & \otimes & & & \vdots \\ \vdots & & & \ddots & & \vdots \\ \vdots & & & & \otimes & 0 \\ x & \dots & \dots & \dots & x & \otimes \end{array}\right] \tag{18}$$

The x entries denote arbitrary elements and the $\otimes$ are nonzero for a controllable system. This transformation uses only numerically stable elementary permutation and elimination steps for the computation of $\boldsymbol{A_H}$, $\boldsymbol{b}_H$, $\boldsymbol{T}_H$ and $\boldsymbol{T}_H{}^{-1}$. In Hessenberg form the last row of the inverted controllability matrix is

$$\boldsymbol{e}_H^T = \left[e_{H1} \;\; 0 \;\; \dots \;\; 0\right] \tag{19}$$

where $1/e_{H1}$ the product of the $\otimes$-elements in equation (18).Then $\boldsymbol{e}_{\boldsymbol{T}}^{\boldsymbol{T}} = \boldsymbol{e}_{\boldsymbol{H}}^{\boldsymbol{T}} \boldsymbol{T}_H$.

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Biographical Sketch

Juergen Ackermann received the Dipl.-Ing. and Dr.-Ing. degrees from the Technical University Darmstadt, the M.S. degree from the University of California, Berkeley, and the "Habilitation" from the Technical University of Munich, Munich, Germany.

Since 1962, he has been with the German Aerospace Research Establishment (DLR), Oberpfaffenhofen, Germany, where he was Director of the Institute of Robotics and Mechatronics from 1974 until his retirement in 2001. He is also Adjunct Professor at the Technical University of Munich. His main research interests are in parametric robust control and vehicle steering applications. He is author of books on "Sampled-data Control Systems" and "Robust Control".

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